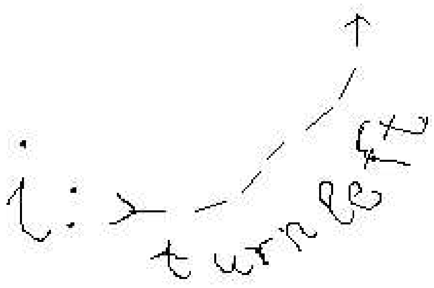
"i i i i i!" cries Jin, the sailor-boy and rubs his cheek. "8 lines port!", shouts the skipper. "8 lines -that's an right angle, so a quarter of a circle - and port-side means turn left, like the stars circle the northpole of the sky. You should never forget: where your cheek is red now, there is red port-side; the other side is green, is star-board."



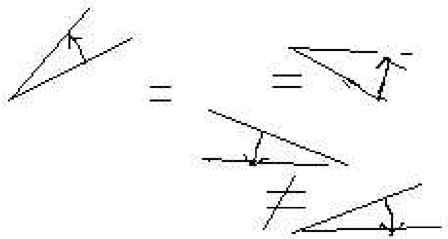
1/4 circle = 90 degrees = 8 lines (when you divided the circle into 32 lines) denotes a
<u>difference of direction = angle</u>.

Rotate to port = left-turn is a
<u>direction of rotating.</u>

Mathematical-analytically is an angle a number;
the direction of rotation add a sign : + ;
right-turn adds a minus-sign. So, we get an

<u>orientated angle, orientated difference of direction.</u>

Geometric-graphically we have to differ the start-side of an angle from it's end-side.



The skipper hands the steer to the helmsman and takes Jin to the map, where he marks the distance they have gone. Jin looks at a line, done by pencil, starting from the harbour. "What can you tell me about ?", asks the skipper and points to the other end of the line. "Before the turn we have gone direction EastNorthEast," says Jin und the skipper adds: " and in the turn itself the mast was the axis, the direction of rotation was to port and the angle was 8 lines." - "ENE before, change to direction NorthNorthWest, the difference between these directions is an right angle, so

ENE + 8' = NNW .''adds Jin.

"And what's the use oft it?" asks the skipper and answers for himself: "The ruler lies here in the point of turn and along-side the direction of our new route." - " We are going to hit the sandbank", interrupts Jin. "Exact, we can look into future, or in plain words, we plan. Three miles more and have to change course."

The course, you take, the direction of drive, is a direction of space and it relates to a basic direction (on maps this is north= on top, and in math's it's the direction to the right hand = east).
You can notate direction by a name, like SouthEast, or by a number, like the 30st degree.
(just like a number of a house, it's a number in an order).

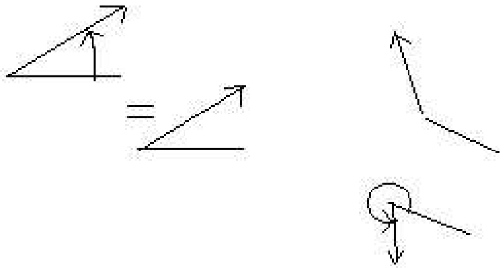
On the map, we can put down the compass anywhere, all directions slide parallel with this and so they don't change in relation to the basic direction.

An orientated angle we can relate not only to the basic direction.Taking a turn (that's a rotation, where the axis is mobil and not fixed to a location) has the direction of driving as reference direction, as beginning side and the new direction of driving as end-side.

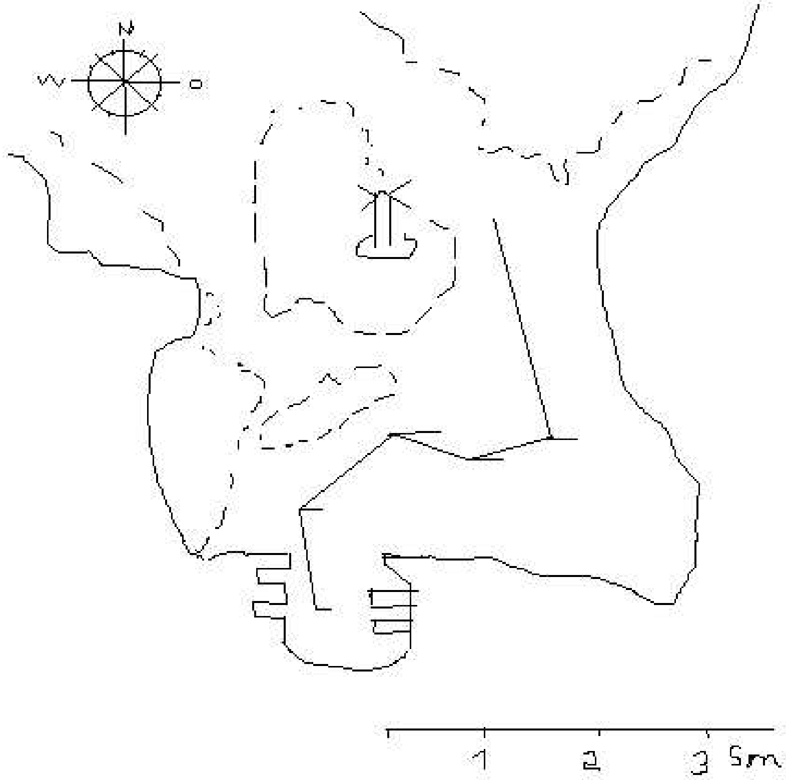
"You draw every line of driving with length and direction?" aks Jin. The answer was "Yes.". "In the harbour they showed me the radar and onscreen the location of any ship was given by a length, that' s the distance from the harbour and by direction."
"We could do the same - look, here in the atlas is a map of the northpole-region. This is the basic direction pointing to Greenwich and when you go around the pole, the direction of view from the pole changes. The second kind of moving, driving to the pole or away from it, changes the distance.

"Funny, isn' t it?", Jin says, " what seems to us as going straight - seems to be a rotation and at the same time an enlargement of the distance, looking from the harbour!"

Mathematic-analytically the vectors of a plane are ordered pairs of numbers with certain operations. Using them, you can present lots of things: speed, temperature, forces,...
Graphic-geometrically is a line of driving with length and direction one example of a vector - more general it has a directed length (a length with a + or - sign) and a orientated difference of direction.
It is independent of basic direction, location, unit of scale and coordinate system.



We draw a vector by start-direction and in a left-turning angle to it an end-direction of a certain length and tipped with an arrowhead.
Choosing one location, let' s say the harbour, as a starting point(origin) or point of relation, then every connection from it to a ship is a location vector or position, it has a length and an angle to the basic direction.
If we bind a vector (f.e. 3 sm, direction NNW) to a location, f.e the turning point, we get here a field-vector-arrow, related to the basic direction.



"Something like that we could need now: we have to get round the light-house and enlarge the distance from it. Draw a line from the light-house to the ship, and a second one from the light-house to the place, whereto we want to go - both have a difference in angle and the second line is longer than the first."

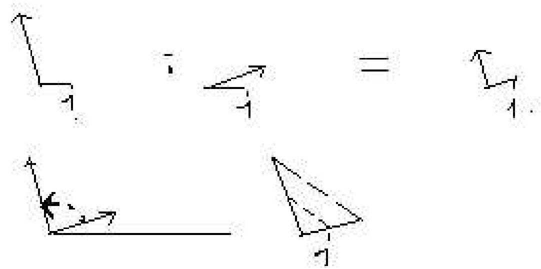
The connection of arrows, the notch of the second to the head of the first,, with both start-directions parallel, we call addition.

By this we get the route, a movement from notch to head and furtheron to the next head. When we add an vector of location from the origin or when we start from it, we get the locations, respectively the movement of the location vector alongside the route.

The difference between two location vectors is the movement from the head of the first to the head of the second, is the route as an arrow.Calculate by: later minus earlier.

Drawing is quite simple for this, calculating is much more difficult, just look at the angles and the length.

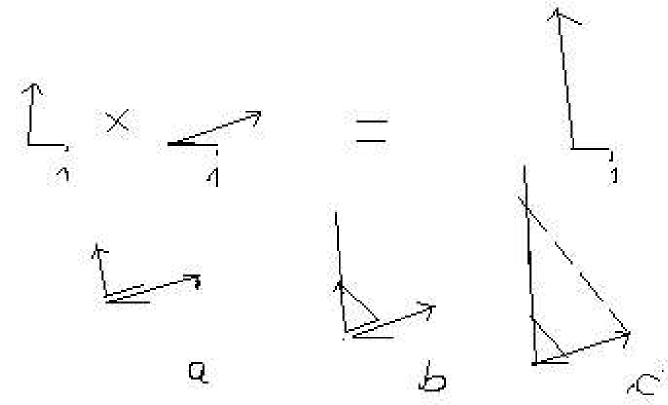
More simple is calculatimg a division and a multiplication of vectors.The field-vectors from the lighthouse to the ship before and after rounding it, both start at the same point and both have the same start-direction - the proportion of the two lengths, together with the difference of angle is a vector too, the result of the division.



(s , ß) : (r , @) := (s / r , ß - @)

Jin answers : " It' s opposite to the turn. There we add an angle to a direction, to get the new direction. Here we'   ve the difference of two directions as an angle."
" Aye," the skipper said, " but now we have to go safe around the lighthouse!"

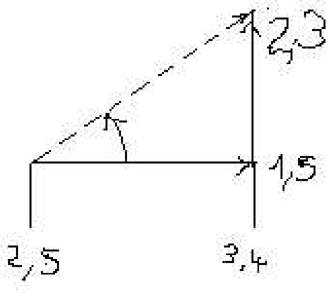
In the turning-point we rotated a vector of our course , of indeterminated length r and of direction NNE or 22.5 degrees from east to the left by a right angle, without changing length, so multiplied by (1 , 90°):
(1 , 90°) times (r , 22.5°) = (r , 22.5°+90°)
and for this , the starting-side of the second angle had to coincide with the end-side of the first, and both had the same axis.
For rounding the lighthouse we can multiply the fieldvector lighthouse-ship with a vector (1.5 , 88°), which will enlarge the fieldvector and turn it into the new fieldvector lighthouse-destination:
 (1.5 , 88°) times (2 , 25°) = (3 , 113°).



(r , @) times (s , ß) = (r times s , @ + ß).

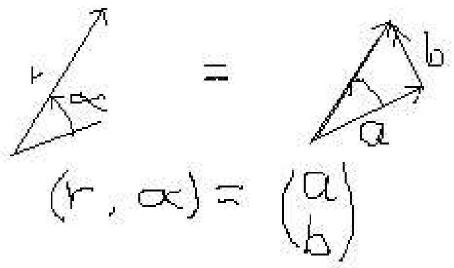
When we install a basic direction we can omit the start-side of all vectors, which start-side points in this basic-direction. So, we can draw a vector as a (one-legged) arrow.

After passing the lighthouse, Jin asks:"On the northpole-map all locations were fixed by their distance from the pole and their angle to the 0°-line. On our map there' s no pole for the locations." - "Good observation", on this map you see a right-angled grid, serving the same purpose.Let' s take this point as starting-point for everything on the map.The distance between two grid-points is always one sea-mile. To get to this point of our route, here, close to the harbour, we go 2.5 sm to the right and 1.5 sm up. These numbers are called the coordinates of the place.Now read this next point." - "3.4 sm to the right and 2.3 sm up." The skipper draws on a paper a triangle:

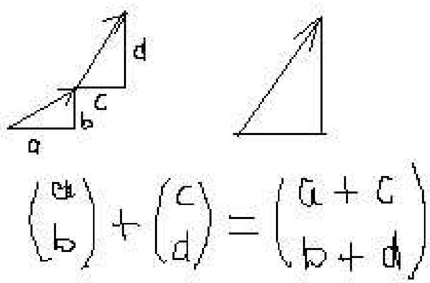


"Now you can measure the length of the route and the angle too." Jin says:"Now i could represent the route too by two rightangled numbers, just like the coordinates of the places."But beware of steering such a rightangled course!" the skipper is kidding. "Eh, yes, sometimes ... when you get a slap..."
But the skipper wants to keep the last word:"Did we choose a different grid-point for a start, even one, that'   not on the map, then you only had to subtract his coordinates from that of all places."

Graphic-geometrically every vector can be represented by an orientated length and an orientated difference of direction. This we call the polar form - the same vector also has a cartesian form:
two directed length, the second one in an right angle by left-turn to the first one.

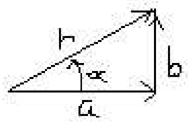


Up to now addition and subtraction were more than difficult to calculate, it changes to simple:



(the starting sides have to be parallel for this).

The conversion cartesian - polar is not difficult, when you can draw rightangled triangles.
Calculating it' s more, but so exact, as you want to:



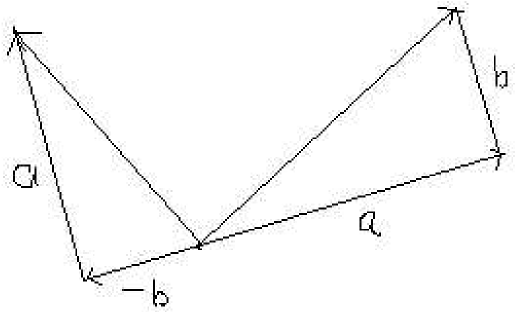
$$(r, \alpha) = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \begin{pmatrix} r \cdot \cos\alpha \\ r \cdot \sin\alpha \end{pmatrix}$$

and the other way round, we calculate from a and b:

$r = \sqrt{a^2 + b^2}$ and
@ = arc cos (a / r) , if $b > 0$ and $b=0$
@ = 360° - arc cos (a / r) , if $b < 0$
and now we can add vectors in polar form too:
converse into cartesian form, add them and
reverse into polar form.

But Jin doesn' t give up:"And if you enlarge the route, let' s say by threefold, then both components will be threefold too:"
"Aye.",the skipper said, " and if, and if - okay, turn by 8 lines, i mean an right angle to the left."
"The first component now shows to top, and the second is inline with the first, but in opposite direction.



This order, turn port-side by 8 lines, i call simply i.

$$i \text{ times} \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} -b \\ a \end{pmatrix}$$

"Wonderful," the skipper said, "now you know everything about planes. But, but our earth is a sphere, a ball. We simplified. The coordinate-system on the northpole-map and on our map are one and the same - think about that."

For the <u>multiplication of vectors in cartesian form</u> we dont have to make a detour to the polar form, though multiplication was easy there.By Jin' s operation

i = turn by 90°

$$i \cdot \begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} -b \\ a \end{pmatrix}$$

you get: $\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$

Deduction:

The number 1 as a factor doesn't change anything, in polar form (1 , 0°) and in cartesian form

$1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

A number c forms, it stretches or it shrinks

$c \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$ and $c \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix}$

$i \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = i\,i = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ $\quad d\,i = \begin{pmatrix} 0 \\ d \end{pmatrix}$

and finally $\begin{pmatrix} c \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$.

The rules of computation, we are used to, stay on and we can use everything together:

$$(3+4i)\,(-1+i) + 3i \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 6 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \end{pmatrix} = 0$$

For those, who can calculate vectors a little bit:

You know vectors of a plane. how to add, to stretch and to shrink them. In this stinky normal plane you can rotate vectors too. The one and only necessity for this is a new multiplication:

$$\begin{pmatrix} a \\ b \end{pmatrix} \text{mal} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ ad + bc \end{pmatrix}$$

(This is neither the dot-product, nor the cross-product - these are not necessary here).

You can represent a vector by length r and angle @,
a = r cos @ and b = r sin @. With it you rotate the second vector by @ and form (stretch/shrink) it by the factor r.
Some might know , that you can rotate by matrices too:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{or} \quad r \begin{pmatrix} \cos @ & -\sin @ \\ \sin @ & \cos @ \end{pmatrix}$$

rotates by @ and forms with factor r

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}$$

And that's that !

This road gone our harvest is a new way of writing vectors, we can add to the previous ones.Every vector can be written as a sum of two basic-vectors, both formed (stretched/shrinked):

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{We call} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} := i$$

$$\text{Then} \quad \begin{pmatrix} a \\ b \end{pmatrix} = a + i\,b\,. \text{ And } \; i \text{ times} \begin{pmatrix} c \\ d \end{pmatrix} := \begin{pmatrix} -d \\ c \end{pmatrix}.$$

Now only one rotation is left, it' s called i.
It' s an operation turn left by an right angle = exchange the positions of c and d and multiply the new first number by -1.
Two turns result in i times i = - 1.
Everything else is stretching addition,...

This simple way of writing 2D (in a bit different form)and the rules for calculating were developed 1572 by Bombelli, and we should give this his name. It was the first calculation with vectors!
Bombelli was engineer for hydraulics in northern Italy, a world of merchandise, inventing the dopik (double book-keeping), and this at a time, when Mercator introduced the coordinates of maps.

26.10.2003 added to page 5:"Calculate by: later minus earlier."