First Part of

ON THE ANALYTICAL REPRESENTATION OF DIRECTION AN ATTEMPT APPLIED CHIEFLY TO THE SOLUTION OF PLANE AND SPHERICAL POLYGONS (BY CASPAR WESSEL, SURVEYOR)

This present attempt deals with the question, how may we represent direction analytically; that is, how shall we express right lines so that in a single equation involving one unknown line and others known, both the length and the direction of the unknown line may be expressed.

To help answer this question I base my work on two propositions which to me seem undeniable. The first one is: changes in direction which can be effected by algebraic operations shall be indicated by their signs. And the second: direction is not a subject for algebra except in so far as it can be changed by algebraic operations. But since these cannot change direction (at least, as commonly explained) except to its opposite, that is, from positive to negative, or *vice versa,* these two are the only directions it should be possible to designate, by present methods; for the other directions the problem should be unsolvable. And I suppose this is the reason no one has taken up the matter. (Unless it be Magister Gilbert, in Halle, whose prize memoir on *Calculus Situs* possibly contains an explanation ofthis subject.) It has undoubtedly been considered impermissible to change anything in the accepted explanation of these operations.

And to this we do not object so long as the explanation deals only with quantities in generaL But when in certain cases the nature of the quantities dealt with seems to call for more precise definitions of these operations and these can be used to advantage, it ought not to be considered impermissible to offer modifacations For as we pass from arithmetic to geometric analysis, or from operations with abstract numbers to those with right lines, we meet with quantities that have the same relations to one another as numbers, surely; but they also have many more. If we now give these operations a wider meaning, and do not as hitherto limit their use to right lines of the same or opposite direction; but if we extend somewhat our hitherto narrow concept of them so that it becomes applicable not only to the same cases as before, but also to infinitely many more; l say, if we take this liberty, but do not violate the accepted rules of operations, we shall not contravene the first law of numbers. We only extend it, adapt it to the nature of the quantities considered, and observe the rule of method which demands that we by degrees make a difficult principle intelligible.

It is not an unreasonable demand that operations used in geometry be taken in a wider meaning than that given to them in arithmetic. And one will readily admit that in this way it should be possible to produce an infinite number of variations in the directions of lines. Doing this we shall accomplish, as will be proved later, not only that all impossible operations can be avoided - and we shall have light on the paradoxical statement that at times the possible must be tried by impossibble meansbut also that the direction of all lines in the same plane can be expressed as analytically as their lengths without burdening the mind with new signs or new rules. There is no question that the general validity of geometric propositions is frequently seen with greater ease if direction can be indicated analytically and governed by algebraic rules than when it is represented by a figure, and that only in certain cases. Therefore it seems not only permissible, but actually profitable, to make use of operations that apply to other lines than the equal (those of the same direction) and the opposite. On that account my aim in the following chapters will be:

1. First, to define the rules for such operations;
2. Next, to demonstrate their application when the lines are in the same plane, by two examples;
3. To define the direction of lines lying in different planes by a new method of operation, which is not algebraic;
4. By means of this method to solve plane and spherical polygons;
5. Finally to derive in the same manner the ordinary formulae of spherical trigonometry.

These will be the chief topics of this treatise. The occasion for its being was my seeking a method whereby I could avoid the impossible operations; and when I had found this, I applied it to convince myself of the universality of certain well-known formulae. The Honourable Mr Tetens, Councillor-of-State, was kind enough to read through these first investigations. It is due to the encouragement, counsel and guidance of this distinguished savant that this paper is minus some of its first imperfections and that it has been deemed worthy to be included among the publications of the Royal Academy.

A METHOD WHEREBY FROM GIVEN RIGHT LINES TO FORM OTHER RIGHT LINES BY ALGEBRAIC OPERATIONS: AND HOW TO DESIGNATE THEIR DIRECTIONS AND SIGNS

Certain homogeneous quantities have the property that if they are placed together, they increase or diminish one another only as increments or decrements.

There are others which in the same situation effect changes in one another in innumerable other ways. To this class belong right lines.

Thus the distance of a point from a plane may be changed in innumerable ways by the point describing a more or less inclined right line outside the plane.

For, if this line is perpendicular to the axis of the plane, that is, if the path of the point makes a right angle with the axis, the point remains in a plane parallel to the given plane, and its path has no effect on its distance from the plane.

If the described line is indirect, that is, if it makes an oblique angle with the axis of the plane, it will add to or subtract from the distance by a length less than its own; it can increase or diminish the distance in innumerable ways.

If it is direct, that is, in line with the distance, it will increase or diminish the same by its whole length; in the first case it is positive, in the second, negative.

Thus, all the right lines which can be described by a point are, in respect to their effects upon the distance of a given point from a plane outside the point, either direct or indirect or perpendicular ('Indifferent' would be a more fitting name were it not so unfamiliar to our ears.) according as they add to or subtract from the distance the whole, a part, or nothing, of their own lengths.

Since a quantity is called absolute if its value is given as immediate and not in relation to another quantity, we may in the preceding definitions call the distance the absolute line; and the share of the relative line in lengthening or shortening the absolute line may be called the 'effect' of the relative line.

There are other quantities besides right lines among which such relations exist. It would therefore not be a valueless task to explain these relations in general, and to incorporate their general concept in an explanation on operations. But l have accepted the advice of men of judgement, that in this paper both the nature of the contents and plainness of exposition demand that the reader be not burdened here with concepts so abstract. l shall consequently make use of geometric explanation only. These follow.

§ 1

Two right lines are added if we unite them in such a way that the second line begins where the first one ends, and then pass a right line from the first to the last point of the united lines. This line is the sum of the united lines.

For example, if a point moves forward three feet and backward two feet, the sum of these two paths is not the first three and the last two feet combined; the sum is one foot forward. For this path, described by the same point, gives the sarne effect as both the other paths.

Similarly, if one side of a triangle extends from a to b and the other from b t o c, the third one from a to c shall be called the sum. We shall represent it by $ab+bc$, so that ac and $ab+bc$ have the same meaning; or $ac=ab+bc=-ba+bc$, if ba is the opposite of ab. If the added lines are direct, this definition is in complete agreement with the one ordinarily given. If they are indirect, we do not contravene the analogy by calling a right line the sum of two other right lines united, as it gives the same effect as these. Nor is the meaning I have attached to the symbol + so very unusual; for in the expression $ab+ba/2=\frac{1}{2}ab$ it is seen that $ba/2$ is not a part of the sum. We may therefore set $ab+bc=ac$ without, on that account, thinking of bc as a part of ac; $ab+bc$ is only the symbol representing ac.

§2

If we wish to add more than two right lines we follow the same procedure. They are united by attaching the terminal point of the first to the initial point of the second and the terminal point of this one to the initial point of the third, etc. Then we pass a right line from the point where the first one begins to the point where the last one ends; and this we call their sum.

The order in which these lines are taken is immaterial; for no matter where a point describes a right line within three planes at right angles to one another, this line has the same effect on the distances of the point from each of the planes. Consequently any one of the added lines contributes equally much to the determination of the position of the last point of the sum whether it have first, last, or any other place in the sequence. Consequently, too, the order in the addition of right lines is immaterial. The sum will always be the same; for the first point is supposed to be given and the last point always assumes the same position.

So that in this case, too, the sum may be represented by the added lines connected with one another by the symbol +. In a quadrilateral, for example, if the first side is drawn from a to b, the second from b to c, the third from c to d, but the fourth from a to d, then we may write: $ad = ab+bc+cd$.

§3

lf the sum of several lengths, breadths and heights is equal to zero, then is the sum of the lengths, the sum of the breadths, and the sum of the heights each equal to zero.

§4

lt shall be possible in every case to form the product of two right lines from one of its factors in the same manner as the other factor is formed from the positive or absolute line set equal to unity. That is:

Firstly, the factors shall have such a direction that they both can be placed in the same plane with the positive unit.

Secondly, as regards length, the product shall be to one factor as the other factor is to the unit. And,

Finally, if we give the positive unit, the factors, and the product a common origin, the product shall, as regards its direction, lie in the plane of the unit and the factors and diverge from the one factor as many degrees, and on the same side, as the other factor diverges from the unit, so that the direction angle of the product, or its divergence from the positive unit, becomes equal to the sum of the direction angles ot the factors.

§**5**

Let +l designate the positive rectilinear unit and +€ a certain other unit perpendicular to the positive unit and having the same origin; then the direction angle of +1 will be equal to 0°, that of - l to 180°, that of +€ to 90°, and that of - € to-90° or 270°. By the rule that the direction angle of the product shall

equal the sum of the angles of the factors, we have :
(+l)(+l)=+1; (+1) (-1)= -1; (-l)(- l)=+l; (+1) (+€)=+€;
(+l)(- €)= - €; (-1) (+€)= - €;(- l)(- €)=+€; (+€) (+€)= - l;
(+€)(- €)=+l;(- €)(- €)=- l.
From this it is seen that c is equal to $\sqrt{-1}$; and the divergence of the product is determined such that not any of the common rules of Operation are contravened.

§6

The cosine of a circle arc beginning at the terminal point of the radius +l is that part of the radius, or of its opposite, which begins at the centre and ends in the perpendicular dropped from the terminal point of the arc. The sine of the arc is drawn perpendicular to the cosine from its end point to the end point of the arc.
Thus, according to § 5, the sine of a right angle is equal to $\sqrt{-1}$.

Set $\sqrt{-1}$ = €. Let *v* be any angle, and let sin *v* represent a right line of the same length as the sine of the angle *v,* positive, if the measure of the angle terminates in the first semi-circumference, but negative, if in the second. Then it follows from §§ 4 and 5 that € sin *v* expresses the sine of the angle *v* in respect of both direction and extent.

§7

In agreement with §§ l and 6, the radius which begins at the centre and diverges from the absolute or positive unit by angle *v* is equal to cos v+€ sin *v.* But, according to § 4, the product of the two factors, of which one diverges from the unit by angle *v* and the other by angle *u,* shall diverge from the unit by angle *v+u.* So that if the right line cos v +€ sin v is multiplied by the right line cos *u*+€sin *u,* the product is a right line whose direction angle is *v+u*. Therefore, by §§ l and 6, we may represent the product by cos (*v+u)* + €sin *(v+u*).

§8

The product (cos *v*+€ sin *v)* (cos *u*+€ sin *u),* or cos *(v + u)+* € sin *(v + u*), can be expressed in still another way, namely, by adding into one sum the partial products that result when each of the added lines whose sum constitutes one factor is multiplied by each of those whose sum constitutes the other. Thus, if we use the known trigonometric formulae.
cos *(v+u)=*cos *v* cos *u*—sin *v* sin *u,*
sin *(v+u)==cos v* sin *u*+cos *u* sin *v,*
we shall have this form:
(cos *v*+€ sin *v*) (cos *u*+€ sin u*)* = cos *v* cos *u* - sin *v* sin *u*
+€ (cos *v* sin *u* - *cos u* sin *v).*
For the above two formulae can be shown, without great difficulty, to hold good for all cases - be one or both of the angles acute or obtuse, positive or negative. In consequence, the propositions derived from these two formulae also possess universality.

§9

By § 7 $\cos v + € \sin v$ is the radius of a circle whose length is equal to unity and whose divergence from $\cos 0°$ is the angle v. lt follows that $r \cos v + r€ \sin v$ represents a right line whose length is r and whose direction angle is v. For if the sides of a right-angled triangle increase in length r times, the hypotenuse increases r times; but the angle remains the same. However, by § l, the sum of the sides is equal to the hypotenuse; hence,
$r \cos v + r € \sin v = r (\cos v + € \sin v)$.
This is therefore a general expression for every right line which lies in the same plane with the lines $\cos 0°$ and $€ \sin 90°$, has the length r, and diverges from $\cos 0°$ by v degrees.

§10

If a, b, denote direct lines of any length, positive or negative, and the two indirect lines $a+€ b$ and $c+€ d$ lie in the same plane with the absolute unit, their product can be found, even when their divergences from the absolute unit are unknown. For we need only to multiply each of the added lines that constitute one sum by each of the lines of the other and add these products; this sum is the required product both in respect to extent and direction: so that $(a+€ b) (c+€ d)=ac—bd+ €(ad+bc)$.
Proof'. Let the length of the line $a +€ b$ be A, and its divergence from the absolute unit be v degrees, also let the length of $c- € d$ be C, and its divergence be u. Then, by § 9, $a+€ b=A \cos v+A€ \sin v$, and $c+€d=C \cos u+C €\sin u$. Thus $a=A \cos v$, $b=A \sin v$, $c=C \cos u$, $d=C \sin u$ (§ 3). But, by § 4, $(a+€b) (c+€d)=$
$AC[\cos (v +u) +€ \sin (v +u)]=AC [\cos v \cos u— \sin v \sin u +€ (\cos v \sin u+\cos u \sin v)]$ (§ 8). Consequently, if instead of $AC \cos v \cos u$ we write ac, and for $AC \sin v \sin u$ write bd, etc., we shall derive the relation we set out to prove.
lt follows that, although the added lines of the sum are not all direct, we need make no exception in the known rule on which the theory of equations and the theory of integral functions and their simple divisors are based, namely, that if two sums are to be multiplied, then must each of the added quantities in one be multiplied by each of the added quantities in the other. lt is, therefore, certain that if an equation deals with right lines and its root has the form $a+€b$, then an indirect line is represented. Now, if we should want to multiply together right lines which do not both lie in the same plane with the absolute unit, this rule would have to be put aside. That is the reason why the multiplication of such lines is omitted here. Another way of representing changes of direction is taken up later, in §§ 24-35.

§ 11

The quotient multiplied by the divisor shall equal the dividend. We need no proof that these lines must lie in the same plane with the absolute unit, as that follows directly from the definition in § 4. lt is easily seen also that the quotient must diverge from the absolute unit by angle $v—u$, if the dividend diverges from the

same unit by angle v and the divisor by angle u.
Suppose, for example, that we are to divide A (cos v+€ sin v) by B (cos u+€ sin u). The quotient is

$$\frac{A}{B}[\cos(v-u)+€\sin(v-u)]$$

since

$$\frac{A}{B}[\cos(v-u)+€\sin(v-u)] \times B(\cos u+€\sin u)=A(\cos v+€\sin v),$$

by § 7. That is, since $\frac{A}{B}[\cos(v-u)+€\sin(v-u)]$ multiplied by the divisor B (cos u +€ sin u) equals the dividend A (cos v +€ sin v), then $\frac{A}{B}[\cos(v-u)+€\sin(v-u)]$ must be that required quotient.

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§ 12

When a, b, c and d are direct lines, and the indirect a + € b and c + € d are in the same plane with the absolute unit, then
$1 / (c+ €d) = (c + €d) / (c^2 + d^2)$, and the quotient
$(a + €b) : (c + €d) = (a + €b)*1/(c + €d) =$
$(a + €b) * (c - €d)/(c^2 + d^2)$,
as following § 9 one can replace
$a + €b=A(\cos v + € \sin v)$ and
$c + €d=C(\cos u + € \sin u)$ and therefore $c- €d=C(\cos u - €\sin u)$
according to §3, and as $(c+ €d)(c - €d)=c^2+ d^2 =C^2$ (§10)
follows $(c - €d) / (c^2 + d^2) = 1/C (\cos u - €\sin u)$, §10, or
$(c- €d) / (c^2 + d^2) = 1/C (\cos (- u) + € \sin (- u) =$
$1 / (c + €d)$, §11, this multiplied with
$a + €b=A(\cos v +€\sin v)$, delivers
$(a + €b)*(c - €d) / (c^2 + d^2) =$
$A/C (\cos(v - u) + €\sin (v - u)) = (a + €b)/(c + €d)$, §11.

Indirect quantities of this kind have in common with the direct ones, that, when the dividend is a sum of several quatities, then these deliver, when divided with the divisor, several quotients, the sum of those is the quotient, one is looking for.

§ 13

When m is an integer, then $\cos v / m + €\sin v / m$ produces a power of $\cos v + €\sin v$ (§7);
therefore $(\cos v + € \sin v) ^{1/m} = \cos v / m + €\sin v/m$;
but from § 11 $\cos(- v / m) + €\sin(- v / m)=1/(\cos v / m + €\sin v / m)=$
$1/(\cos v +€\sin v) ^{1/m}=(\cos v + €\sin v) ^{(- 1/m)}$. . So we always have, no matter if m is positive or negative, $\cos v / m + €\sin v / m =$ $(\cos v + € \sin v) ^{1/m}$ and therefore, when m and n are both integers $(\cos v + € \sin v) ^{1/m} = \cos v / m + €\sin v / m$.

From this one can find the value of terms like

$$\sqrt[n]{b + c\sqrt{-1}}$$ or

$$\sqrt[m]{a + \sqrt[n]{b + c\sqrt{-1}}}$$

for example

$$\sqrt[3]{4\sqrt{3} + 4\sqrt{-1}}$$ denotes a straight line, which is of length 2 and which has an angle of 10° with the absolute unit.

§14

When two angles share the same sinus and the same cosinus, then their difference is either = 0, or - / + 4 right angles, or a multitide of + / - 4 right angles and reversed, when the difference between two angles is 0 or + / - 4 right angles, taken one or more times, then their sinus, as well as their cosinus, is equal.

§ 15

When m is an intger and pi = 360°, then (cos *v* + € sin *v*) ^ 1/m only takes the following m different values
cos *v* / m + €sin *v* / m, cos ((pi + *v*) / m) + €sin ((pi + *v*)/ m),
cos ((2 pi + *v*) / m) + €sin ((2 pi + *v*)/ m),...
cos (((m – 1)pi + *v*) / m) + €sin (((m – 1)pi + *v*)/ m),
as the numbers, pi was mulitplied to in the foregoing sequence, are in an arithmetic progression 1, 2, 3, 4, ..., m – 1. Therefore the sum of any two = m, when one is as far apart from 1 as the other from m – 1, and when their number are odd, then two times the passage to their middle = m, therefore, when one adds ((m – n)pi + *v*)/ m to ((m – *u*)pi + *v*)/ m, and the last one is as much away from (pi + *v*)/ m in the sequence, as ((m – n)pi + *v*)/ m from ((m – 1)pi + *v*)/ m, then their sum is = ((2m - *u* - n)/m) pi + 2 *v* / m = pi +2 *v* / m. But to add (m – n) pi / m is the same, as to subtract (m – n)(- pi) / m; and as the differnce is pi, ((m – n)(- pi)+ *v*) / m has the same cosinus and sinus as ((m – *u*)pi + *v*) / m, according to §14 ; just as ((m - *u*)(- pi)+*v*) / m und (m – n) pi+*v*) / m have the same sinus and cosinus; therefore – pi does not deliver a different value as pi. But that none of these are equal follows from the fact, that the difference between two angles of the sequence always is smaller than pi and never = 0. Neither one finds more values, when continuing the sequence, as then one gets the angles pi + *v* /m, pi + (pi +*v*)/m, pi + (2pi+*v*)/m, and so forth, so according to § 14 the values of their cosinusand sinus are the same as before.In case the angles would drop out of the sequence the numerator of pi wouldn't be multiplied with an integer, and the angles, taken m-fold, couldn't produce an angle, which, when subtracted from

v, resulted in 0 or + / - pi, or a multitude of + / - pi; Therefore the m-th power of such an angle could have a cosinus and sinus = cos *v* + €sin *v*.

§ 16

Without knowing the angle, which an indirect line 1 + x takes to the absolute, one finds, when the length of x is smaller than 1, the power $(1+x)^m = 1+mx/1 + m/1 * (m-1)/2 * x^2 +$ and so forth and when this series is rearranged according to the powers of m, it keeps it's value and will be changed into $1 + m\,l/1 + m^2 l^2/1*2 + m^3 l^3/1*2*3 +$ and so forth, where $l = x - x^2/2 + x^3/3 - x^4/4 +$ and so forth, and is a sum

of a direct line a and a perpendicular b $\sqrt{-1}$, then b is the smallest factor of the angle, which 1 + x has with + 1, and when setting $1+1/1+1/1*2+1/1*2*3+$ and so forth =e , then $(1+x)^m$ or $1+m\,l/1+m^2 l^2/1*2 + m^3 l^3/1*2*3+$ and so forth are denoted with

$e^{(ma + mb\sqrt{-1})}$, that is $(1+x)^m$ has the length e^{ma} and an angle of direction, the factor of which is mb, m shall be presumed positive or negative

So the direction of lines in the same plane can expressed in a different manner too, namely with the help of the natural logarithm. I will bring forward a complete proof of these sentences at a different time, if permitted. Now, as i have given account of, how one has given the sum, the product, the quotient and the power of straight lines, i will give a few examples of the application of this method.

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This translation of "Om Directionens analytiske Betegning..." is an extension of the translation from Professor Martin A. Nordgaard, St Olaf College, Northfield, Minnesota from § 12 onwards, done by myself. Actually i did a translation of this first part into german as well. Both translations can be looked up here:
http://insel.heim.at/mainau/331839/theart.html
(Any correction of my translation is welcome).

The second part of Caspar Wessel's text is about applications to 3D.

Up to my knowledge, there exists no other online-version of Wessel's text, not even in danish, except for the complete version in french:
<http://gallica.bnf.fr/scripts/ConsultationTout.exe?O=N099681>

The first mentioning of these ideas are from 1787 in a report about his advanced methods of geodesic surveying.
Otto B. Beeken has a facsimile of it for us here:
http://home.hia.no/~ottob/Vi%E8te-Wessel.pdf

Caspar Wessel wrote his text in1797 and published 1799, in the time of the french revolution.

1806ff Argand and others published similar ideas
http://gallica.bnf.fr/ark:/12148/bpt6k110283x

1814 Cauchy called certain functions from R2 to R2
"monotypique", Briot and Bouquet later called these "holomorphic".

1834 resp 1837 William R. Hamilton published
"Theory of Conjugate Functions or Algebraic Couples.."
David R. Wilkins made this available to us:
http://www.maths.tcd.ie/pub/HistMath/People/Hamilton

In this context it might be of interest, that Gauss never mentioned Caspar Wessel (and Hamilton neither, up to my knowledge). In 1822 Gauss won the Copenhagen University Prize. Gauss wrote about complex numbers in 1831 a text “Theoria residuorum biquadraticorum“. And Gauss became a geographic surveyor, just like Wessel a 'Landmaaler', and did the geodesic survey of the state of Hannover. But by this time Oldenburg in Oldenburg, not in Schleswig-Holstein, has changed from danish posession into the kingdom of Hannover. Now, Wessel had done a geodesic survey of Oldenburg, as he was in charge of the surveying of all of Denmark, and one can speculate if Gauss learned from his advanced method in surveying, and if Gauss made use of his survey of this part of Hannover.
http://www-history.mcs.st-and.ac.uk/history/Biographies/Gauss.html
http://www-history.mcs.st-and.ac.uk/history/Biographies/Wessel.html

But still more of interest is the fact that at least up to 1851 no one used in the geometric represention of complex numbers the expresssion “imaginary axis“ (and “real axis“), nowadays so popular.
http://mathworld.wolfram.com/ImaginaryAxis.html
Riemann for example just writes about x-axis and the y-axis
http://www.maths.tcd.ie/pub/HistMath/People/Riemann/Grund/

Gauss coined in 1831 the word 'complex number', as jeff writes in
http://members.aol.com/jeff570/g.html
But Gauss did not saw a need to rename the y-axis. In the contrary, in this “ Theoria residuorum biquadraticorum“ he wrote:
“The pretended difficulties of the theory of the imaginary magnitudes did arise to the major part from little appropiate denotions. If, as the imagination of two dimensions is suggesting to us, we would have called these magnitudes instead of 'positive, negative, imaginary' now 'ahead, backwards, sidewards', so simplicity would have entered in place of confusion, clearness in place of darkness.“(translated by me from this source
http://members.chello.at/gut.jutta.gerhard/imaginaer3.htm

Actually there is no mathematical definition of the imaginary axis. From the time of Wessel on one can talk of the representation on an ordinary or real plane, and from Hamilton on, one can talk of calculating with complex numbers inside the R2, the set of ordered pairs of real numbers. And if you look for a strict definition of C, you only can find, that it is (R2,+,*), a vector-space and commutative field - i stick to this!

But who introduced into math the words
imaginary axis,
complex plane, Gauss-Ebene or Argand diagramm?

Jeff also does not know who invented the denotion
'imaginary axis'. And he mentions this:
“As a way of removing the stigma of the name, the American mathematician Arnold Dresden (1882-1954) suggested that imaginary numbers be called *normal* numbers, because the term "normal" is synonymous with perpendicular, and the y-axis is perpendicular to the x-axis (Kramer, p. 73). The suggestion appears in 1936 in his *An Invitation to Mathematics.*“
Some still might like to be impressive as a master of the imaginary in using the associative power of the word 'imaginary axis'.

Shouldn't we remove this kind of black magic out of math?